THE ASSOCIATED VARIETY OF A POISSON PRIME IDEAL

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ABSTRACT. We prove that the associated variety of a Poisson prime ideal of the centre of a symplectic reflection algebra at parameter $t = 0$ is irreducible.

1. INTRODUCTION

1.1.   The study of the primitive ideals of an algebra is an approximation of its representation theory. One model case where this study has been fruitful is that of the enveloping algebra of a complex semisimple Lie algebra. The history of our result can be traced back to the well known theorem by Borho and Brylinski [1] and by Joseph [8] stating that in the above context, the associated variety of a primitive ideal is irreducible, and in fact that result can be read off as a corollary of our main theorem.

1.2.   The symplectic reflection algebras of Etingof and Ginzburg [5] are an interesting class of algebras with applications to integrable systems, invariant theory and geometry. Their behaviour varies according to a parameter t. When $t \neq 0$ they have trivial centres, while when $t = 0$ they have large centres and the geometry of the centres plays a leading role.

Let $\mathbf{v}$ be an even dimensional complex vector space with symplectic form ω, and let G be a finite subgroup of the symplectic group of $\mathbf{v}$. Let the triple $(\mathbf{v}, \omega, G)$ be indecomposable. The symplectic reflection algebras, $H_{t,\mathbf{c}}$, are deformations of the skew group ring, $\mathbb{C}[\mathbf{v}] * G$, and their spherical subalgebras, $eH_{t,\mathbf{c}}e$, are deformations of $\mathbb{C}[\mathbf{v}]^G$, the ring of G-invariants.

1.3.   It was proved by Ginzburg [6, Theorem 2.1] that when $t \neq 0$ the associated variety of a primitive ideal of $eH_{t,\mathbf{c}}e$ is irreducible. His method was to generalise the Lie theoretic result using the ideas of Poisson geometry and symplectic leaves. We extend this result to the case when $t = 0$. Here the algebra $eH_{0,\mathbf{c}}e$ is commutative and is isomorphic to the centre, $Z_{0,\mathbf{c}}$, of $H_{0,\mathbf{c}}$. In fact, the centre has the structure of a Poisson algebra. It was shown by Brown and Gordon in [2] that it is the Poisson prime ideals of $Z_{0,\mathbf{c}}$ which provide the natural first step in understanding the finite dimensional representation theory of $H_{0,\mathbf{c}}$. Our main result is the following.

The associated variety of a Poisson prime ideal of $Z_{0,\mathbf{c}}$ is irreducible.

In fact, the result we prove (Theorem 2.3) is somewhat more general than this, and includes Ginzburg's result as a special case. Our proof is modelled on Ginzburg's proof of [6, Theorem

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2.1], which in turn is based on a proof by Vogan [11, §3 − 4] of the result for enveloping algebras.

1.4. As discussed in 3.5, our hope is that this will allow some kind of description of irreducible finite dimensional representations of $H_{0,\mathbf{c}}$ by subgroups of G. We hope to study this further in later work.

Our paper is organised as follows. In §2 we introduce basic definitions and state the main theorem; we discuss applications in §3. We prove the main theorem in the remaining sections: §4 states a number of preliminary results which we use in the proof, which is given in §5.

2. The Main Theorem

2.1. **Poisson structures.**

Definition. Let R be an affine commutative $\mathbb{C}$-algebra. We say that R is a Poisson algebra if there exists a non-trivial Poisson bracket $\{-, -\} : R \times R \to R$. That is, $\{-, -\}$ is a nonzero skew-symmetric bilinear map, and for all $x, y, z \in R$, $\{xy, z\} = x\{y, z\} + \{x, z\}y$ and $\{x, \{y, z\}\} = \{\{x, y\}, z\} + \{y, \{x, z\}\}$. We shall say that an affine variety over $\mathbb{C}$ is *Poisson* if its coordinate ring is a Poisson algebra.

Let $(R, \{-, -\}_R)$ and $(S, \{-, -\}_S)$ be Poisson algebras. We say that a map $\psi : R \to S$ is a *Poisson homomorphism* if it is an algebra homomorphism such that for all $x, y \in R$, $\psi(\{x, y\}_R) = \{\psi(x), \psi(y)\}_S$. When $R = \bigoplus_{i=0}^{\infty} R_i$ is a graded Poisson algebra we shall say that $\{-, -\}$ *has degree* l if for all i and j, $\{R_i, R_j\} \subseteq R_{i+j+l}$, and l is the minimal integer for which this is true.

Let R be a Poisson algebra and fix an algebra generating set $\{a_1, \ldots a_k\}$. For a closed point $\mathbf{m} \in \operatorname{Spec} R$ we define the rank of the Poisson structure at $\mathbf{m}$ to be the rank of the matrix $(\{a_i, a_j\} + \mathbf{m}) \in M_k(\mathbb{C})$. It is independent of the choice of generators.

Definition. The *symplectic leaf* $\mathcal{S}(\mathbf{m})$ containing a closed point $\mathbf{m}$ of Spec R is the maximal connected complex analytic manifold in Spec R such that $\mathbf{m} \in \operatorname{Spec} R$ and the rank of each closed point in $\mathcal{S}(\mathbf{m})$ equals the dimension of $\mathcal{S}(\mathbf{m})$.

The symplectic leaves of Spec R are related to certain ideals of R.

Definition. Let I be an ideal of a Poisson algebra R. Then I is a *Poisson ideal* if $\{R, I\} \subseteq I$.

It was shown in [12, Proposition 1.3] that when Spec R is smooth there exists a stratification of Spec R by symplectic leaves. One can extend this as in [2, §3.5] to show that for any Poisson algebra, R, there exists a stratification of Spec R by symplectic leaves. For any symplectic leaf $\mathcal{S}$ ($= \mathcal{S}(\mathbf{m})$ for some closed point $\mathbf{m} \in \operatorname{Spec} R$) the closure, $\overline{\mathcal{S}}$, of $\mathcal{S}$ in Spec R is the closed subset, $\mathcal{V}(P)$, of Spec R whose defining ideal, P, is a Poisson prime ideal of R (see [2, Lemma 3.5]).

If Spec R is a finite union of symplectic leaves, we say that Spec R *has finitely many symplectic leaves* (or Spec R has FMSL). One consequence of this is that then the notion of a symplectic leaf becomes algebraic (as opposed to analytic) in the sense that each leaf is a locally closed subset of Spec R. More precisely, we know from [2, Proposition 3.7] that for any Poisson prime, P, of R the smooth locus of the closed subvariety $\mathcal{V}(P)$ is a symplectic leaf in Spec R. This gives a one-to-one correspondence between symplectic leaves in Spec R and Poisson prime ideals of R: $\mathcal{S} \longleftrightarrow P$ where $\overline{\mathcal{S}} = \mathcal{V}(P)$.

2.2. Let A be a $\mathbb{C}$-algebra. We shall say that a $\mathbb{Z}$-filtration, $\mathcal{F}$, of A is *suitable* when we have:

$$0 = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq A$$

and $\mathcal{F}$ satisfies (a) $\mathcal{F}_i \cdot \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$, (b) $\mathcal{F}_0 = \mathbb{C}$, (c) $\dim_{\mathbb{C}} \mathcal{F}_i < \infty$ for all i, and (d) $\mathrm{gr}^{\mathcal{F}} A$ is an affine commutative $\mathbb{C}$-algebra.

Definition. Let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$. We say that A has a *proto-Poisson bracket with respect to $\mathcal{F}$* if there exists a non-zero skew-symmetric $\mathbb{C}$-bilinear map $\langle -, - \rangle : A \times A \to A$ which satisfies, for all $a, b, c \in A$:

   (1) $\langle ab, c \rangle = a \langle b, c \rangle + \langle a, c \rangle b$.
   (2) $\langle a, \langle b, c \rangle \rangle = \langle \langle a, b \rangle, c \rangle + \langle b, \langle a, c \rangle \rangle$.
   (3) There is an integer d, the *degree* of $\langle -, - \rangle$, such that $\langle \mathcal{F}_i, \mathcal{F}_j \rangle \subseteq \mathcal{F}_{i+j+d}$ for all $i, j \in \mathbb{Z}$, but there exist $i, j \in \mathbb{Z}$ such that $\langle \mathcal{F}_i, \mathcal{F}_j \rangle \nsubseteq \mathcal{F}_{i+j+d-1}$.

If the filtration is clear from the context we shall simply say that $\langle -, - \rangle$ is a proto-Poisson bracket.

Examples. i) Let A be an algebra with suitable filtration $\mathcal{F}$. Suppose that A is generated by $\mathcal{F}_1$ with $\mathcal{F}_i = (\mathcal{F}_1)^i$, and that A is not commutative. Let $\langle -, - \rangle$ equal the commutator bracket on A. Then $\langle -, - \rangle$ is a proto-Poisson bracket on A. The only nontrivial condition to check is (3). For this let d be the integer such that $\langle \mathcal{F}_1, \mathcal{F}_1 \rangle \subseteq \mathcal{F}_{d+2}$ but $\langle \mathcal{F}_1, \mathcal{F}_1 \rangle \nsubseteq \mathcal{F}_{d+1}$. There exists such an integer because A is not commutative. It is easily seen that $\langle \mathcal{F}_i, \mathcal{F}_j \rangle \subseteq \mathcal{F}_{i+j+d}$ for all $i, j \in \mathbb{Z}$.

   ii) Let R be a Poisson algebra with suitable filtration $\mathcal{F}$. Let $\{-, -\}$ be the Poisson bracket on R; if $\{-, -\}$ satisfies condition (3) of the definition then it is a proto-Poisson bracket. In particular, for any Poisson algebra, R, with generating set $\{a_1, \ldots, a_t\}$ we can define a filtration, $\mathcal{F}$, by $\mathcal{F}_i = 0$ for $i < 0$, $\mathcal{F}_0 = \mathbb{C}$, $\mathcal{F}_1 = \mathbb{C}1 + \sum_{i=1}^{t} \mathbb{C}a_i$ and $\mathcal{F}_i = (\mathcal{F}_1)^i$ for $i \geqslant 2$. Then $\mathcal{F}$ is a suitable filtration and the Poisson bracket on R is a proto-Poisson bracket (in particular, condition (3) will hold for some choice of d).

Remark. We note that for a commutative algebra A, the commutator bracket is identically zero and so this is never an example of a proto-Poisson bracket.

We shall say that an ideal I of A is a $\langle -, - \rangle$-*ideal* if $\langle A, I \rangle \subseteq I$. In example i), a $\langle -, - \rangle$-ideal is just an ideal of A; in example ii), a $\langle -, - \rangle$-ideal is a Poisson ideal. The following extension to the present setting of the standard Gabber-Hayashi recipe (see [5, §15], for example) for constructing a Poisson bracket in Example (i) has a routine proof which is left to the reader.

Lemma. *Let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$, and proto-Poisson bracket, $\langle -, - \rangle$ of degree d. For homogeneous elements $x, y \in \mathrm{gr}^{\mathcal{F}} A$ of degree k and l respectively, denote lifts of x and y by $\tilde{x}, \tilde{y} \in A$, that is, $\sigma_k(\tilde{x}) = x$ and $\sigma_l(\tilde{y}) = y$ where σ denotes the principal symbol map. Then*

$$\mathrm{gr}\langle -, - \rangle : \mathrm{gr}^{\mathcal{F}} A \times \mathrm{gr}^{\mathcal{F}} A \to \mathrm{gr}^{\mathcal{F}} A; \quad (x, y) \mapsto \sigma_{i+j+d}(\langle \tilde{x}, \tilde{y} \rangle)$$

defines a Poisson bracket of degree d on $\mathrm{gr}^{\mathcal{F}} A$ when extended linearly.

2.3. We can now state our main theorem.

Theorem. *Let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$, and proto-Poisson bracket $\langle -, - \rangle$. Let $\mathrm{gr}^{\mathcal{F}} A$ have Poisson bracket $\mathrm{gr}\langle -, - \rangle$ and let I be a prime $\langle -, - \rangle$-ideal of A. Suppose $X = \mathrm{Spec}\, \mathrm{gr}^{\mathcal{F}} A$ has FMSL with respect to the Poisson bracket induced on it by $\langle -, - \rangle$. Let $V = \mathcal{V}(\mathrm{gr}^{\mathcal{F}} I)$. Then V is irreducible, and is the closure of a symplectic leaf in X.*

The second claim follows quickly from the first. For suppose that V is irreducible. Since I is a $\langle -, - \rangle$-ideal it can easily be seen that $\mathrm{gr}^{\mathcal{F}} I$ is a Poisson ideal, and therefore $\mathrm{rad}(\mathrm{gr}^{\mathcal{F}} I)$ is also Poisson by [4, 3.3.2]. Hence V is a closed irreducible Poisson subvariety and is the closure of a symplectic leaf in X, as discussed in 2.1. We note that a version of Theorem 2.3 is true with the weaker assumption that V (and not X) has FMSL. Then V is irreducible, but is not necessarily the closure of a symplectic leaf of X.

3. APPLICATIONS

3.1. We get as a corollary to Theorem 2.3 a proof of the result of Borho and Brylinski, and Joseph. There is a detailed account, including the background material required, in [11, §3 − 4].

Corollary 1 ([1], [8])**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathcal{U}(\mathfrak{g})$ denote its enveloping algebra. Then the associated variety of a primitive ideal of $\mathcal{U}(\mathfrak{g})$ is irreducible.*

Proof. Let $\{X_1, \ldots, X_m\}$ be a basis of $\mathfrak{g}$ and let $[-, -]$ denote its Lie bracket. There is suitable filtration, $\mathcal{B}$, on $\mathcal{U}(\mathfrak{g})$ where $\mathcal{B}_1 = \mathfrak{g}$ generates $\mathcal{U}(\mathfrak{g})$ as an algebra and $\mathcal{B}_i = (\mathcal{B}_1)^i$ for all $i \geqslant 1$. Now $\mathrm{gr}\mathcal{U}(\mathfrak{g}) = \mathbb{C}[X_1, \ldots, X_m]$, and the variety $\mathrm{Spec}\, \mathrm{gr}\mathcal{U}(\mathfrak{g})$ can be identified with $\mathfrak{g}^*$. As explained in Examples 2.2 (i), setting $\langle -, - \rangle$ equal to the commutator on $\mathcal{U}(\mathfrak{g})$ defines a proto-Poisson bracket on $\mathcal{U}(\mathfrak{g})$. Therefore there is a Poisson bracket, $\mathrm{gr}\langle -, - \rangle$, on $\mathrm{gr}\mathcal{U}(\mathfrak{g})$. However, since $\mathrm{gr}\mathcal{U}(\mathfrak{g}) = \mathbb{C}[X_1, \ldots, X_m]$, it is clear that $\mathrm{gr}\langle -, - \rangle$ is extended from the Lie bracket on $\mathfrak{g}$, giving the so-called Kostant-Kirillov Poisson bracket. Let P be a primitive ideal of $\mathcal{U}(\mathfrak{g})$, and let Q be a minimal primitive ideal contained in P. Now, $\mathfrak{g}^*$ will never have FMSL, but the Poisson subvariety $\mathcal{V}(\mathrm{gr}Q)$ does (by [11, Theorem 5.8]) , the leaves being the nilpotent coadjoint orbits (see [10, Theorem 14.3.1]). If we now take $A = \mathcal{U}(\mathfrak{g})/Q$ with filtration and proto-Poisson bracket induced from $\mathcal{U}(\mathfrak{g})$, Theorem 2.3 tells us that $\mathcal{V}(\mathrm{gr}P)$ is an irreducible subvariety of $\mathcal{V}(\mathrm{gr}Q)$ and therefore also of $\mathfrak{g}^*$. $\qquad\square$

3.2. Before discussing further applications of Theorem 2.3 we introduce quotient varieties $\mathbf{v}/G$ and describe their symplectic leaves.

Let $(\mathbf{v}, \omega, G)$ be an indecomposable symplectic triple (see [5, §1]) - in particular, $\mathbf{v}$ is an even dimensional $\mathbb{C}$-vector space, ω a symplectic form on $\mathbf{v}$ and G a finite subgroup of the symplectic group of $\mathbf{v}$. Let $\mathbb{C}[\mathbf{v}]$ be the coordinate ring of $\mathbf{v}$ and let $\mathbb{C}[\mathbf{v}] * G$ be the skew group ring. This latter algebra has centre $\mathbb{C}[\mathbf{v}]^G$, the ring of G-invariants of $\mathbb{C}[\mathbf{v}]$. Let $e = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}[G]$, then there is an isomorphism of algebras $e\mathbb{C}[\mathbf{v}] * Ge \cong \mathbb{C}[\mathbf{v}]^G$.

The ring of invariants is a Poisson domain with bracket induced by ω which we denote $\{-, -\}_\omega$. We note that $\mathbb{C}[\mathbf{v}]$ is a graded algebra and that $\{-, -\}_\omega$ has degree -2. The variety $\mathbf{v}/G = \mathrm{Spec}\, \mathbb{C}[\mathbf{v}]^G$ has finitely many symplectic leaves and, moreover, the leaves have been

described in [2, Proposition 7.4]. Let $\pi : \mathbf{v} \to \mathbf{v}/G$ be the orbit map and for $v \in \mathbf{v}$ let G_v denote the stabiliser of v in G. Given a subgroup H of G let $\mathbf{v}_H^o = \{v \in \mathbf{v} : H = G_v\}$. The symplectic leaves of $\mathbf{v}/G$ are the sets $\pi(\mathbf{v}_H^o)$ as H runs through subgroups of G for which $\mathbf{v}_H^o \neq \emptyset$. If H and H' are conjugate subgroups of G then $\pi(\mathbf{v}_H^o) = \pi(\mathbf{v}_{H'}^o)$ so in fact the leaves are in one-to-one correspondence with the conjugacy classes of subgroups of G which occur as the stabiliser of some element of $\mathbf{v}$.

3.3. Symplectic reflection algebras. For details of the following see [5].

The symplectic reflection algebras corresponding to $(\mathbf{v}, \omega, G)$, written $H_{t,\mathbf{c}}$ where $t \in \mathbb{C}$ and $\mathbf{c} \in \mathbb{C}^r$ for some r, are isomorphic, as vector spaces, to $\mathbb{C}[\mathbf{v}] \otimes_{\mathbb{C}} \mathbb{C}[G]$. They are deformations of the skew group ring in the sense that, when they are filtered by putting elements of $\mathbf{v}$ in degree one, and putting $\mathbb{C}[G]$ in degree zero, then the associated graded algebras are isomorphic to $\mathbb{C}[\mathbf{v}] * G$ ([5, Theorem 1.3]). The spherical subalgebras $eH_{t,\mathbf{c}}e$ inherit the filtration, and we denote this filtration by $\mathcal{B}$. Their associated graded algebra is $\mathbb{C}[\mathbf{v}]^G$ (a consequence of e being in degree zero). The algebras $eH_{t,\mathbf{c}}e$ are commutative if and only if t is zero ([5, Theorem 1.6]).

3.4. We derive [6, Theorem 2.1] as a special case of Theorem 2.3. We first require an elementary lemma.

Lemma. *Let R be an affine commutative $\mathbb{C}$-algebra with two Poisson brackets, $\{-,-\}_1$ and $\{-,-\}_2$, such that $\{-,-\}_1 = \lambda\{-,-\}_2$ for some non-zero $\lambda \in \mathbb{C}$. Let $X = \mathrm{Spec}\ R$. Then the symplectic leaves of X with respect to $\{-,-\}_1$ are the same as the symplectic leaves of X with respect to $\{-,-\}_2$. In particular, $(X, \{-,-\}_1)$ has FMSL if and only if $(X, \{-,-\}_2)$ has FMSL.*

Proof. The rank of $\{-,-\}_1$ at any closed point $\mathbf{m}$ of X is equal to the rank of $\{-,-\}_2$ at $\mathbf{m}$, so the lemma follows from the definition of symplectic leaf. $\qquad\square$

Corollary 2. *Let $eH_{t,\mathbf{c}}e$ be the algebra described in 3.3, and let $t \neq 0$. Then for any primitive ideal I of $eH_{t,\mathbf{c}}e$, the variety $\mathcal{V}(\mathrm{gr}^{\mathcal{B}}I)$ is irreducible.*

Proof. The filtration, $\mathcal{B}$, described in 3.3 is a suitable filtration on $eH_{t,\mathbf{c}}e$. Let $[-,-]$ be the commutator bracket, then we claim that this is a proto-Poisson bracket on $eH_{t,\mathbf{c}}e$. The only condition of definition 2.2 which is non-trivial is (3), but this follows from [5, Claim 2.25(i)] (the degree, d, is -2 in this case). Therefore $\mathrm{gr}[-,-]$ is a Poisson bracket of degree -2 and so by [5, Lemma 2.23(i)] there is some non-zero $\lambda \in \mathbb{C}$ such that $\mathrm{gr}[-,-] = \lambda\{-,-\}_\omega$. By Lemma 3.4, $\mathrm{Spec}\ \mathbb{C}[\mathbf{v}]^{\mathrm{G}}$, with Poisson bracket $\mathrm{gr}[-,-]$, has FMSL. We can now apply Theorem 2.3: for any prime ideal I of $eH_{t,\mathbf{c}}e$, $\mathcal{V}(\mathrm{gr}^{\mathcal{B}}I)$ is irreducible. In particular, this is true for any primitive ideal I. $\qquad\square$

We extend this to the case when $t = 0$.

Corollary 3. *Let $A = eH_{0,\mathbf{c}}e$ with filtration $\mathcal{B}$ as in (3.3) and denote its Poisson bracket by $\{-,-\}$. Let I be a Poisson prime ideal of A, then $\mathcal{V}(\mathrm{gr}^{\mathcal{B}}I)$ is irreducible.*

Proof. The filtration $\mathcal{B}$ is suitable. We would like $\{-,-\}$ to be a proto-Poisson bracket, and so, as noted in Examples 2.2 ii), we need to show that condition (3) of Definition 2.2 is satisfied. It can be seen from the construction of $\{-,-\}$ that there is some $l \geqslant 2$ so that

$\{\mathcal{F}_i, \mathcal{F}_j\} \subseteq \mathcal{F}_{i+j-l}$, for all $i, j \in \mathbb{Z}$ but $\{\mathcal{F}_i, \mathcal{F}_j\} \nsubseteq \mathcal{F}_{i+j-(l+1)}$, for some $i, j \in \mathbb{Z}$. By [5, Lemma 2.26], we must have $l = 2$.

It remains to show that Spec $\text{gr}^{\mathcal{B}}A = \mathbf{v}/G$ with Poisson bracket $\text{gr}\{-, -\}$ has FMSL. However, by [5, Lemma 2.23], $\text{gr}\{-, -\} = \lambda\{-, -\}_\omega$ for some $\lambda \in \mathbb{C}^*$. Therefore by 3.2 and Lemma 3.4 , $(\mathbf{v}/G, \text{gr}\{-, -\})$ has FMSL. Hence $\mathcal{V}(\text{gr}^{\mathcal{B}}I)$ is irreducible. $\qquad\square$

3.5.   Let $Z = Z_{0,\mathbf{c}}$ and let $H = H_{0,\mathbf{c}}$. Our objective in proving Theorem 2.3 is to better understand the symplectic leaves of Spec Z. For it was shown in [2, Theorem 4.2] that the symplectic leaves of Spec$Z$ control the finite dimensional representation theory of the corresponding symplectic reflection algebra. In more detail, [2, Theorem 4.2] says that if two closed points $\mathbf{m}$ and $\mathbf{n}$ of Spec Z lie in the same symplectic leaf then $H/\mathbf{m}H$ and $H/\mathbf{n}H$ are isomorphic $\mathbb{C}$-algebras.

By [2, Theorem 7.8], Spec Z has FMSL. Therefore we can restate our goal as finding a description of the Poisson prime ideals of Z. By 3.2, the corollary below allows us to attach to each Poisson prime of Z, a conjugacy class of a certain subgroup of G.

3.6.   **The case $\mathbf{t} = \mathbf{0}$:** Let Z and H be as in 3.5. The algebras $eH_{0,\mathbf{c}}e$ and Z are both Poisson algebras via the Gabber-Hayashi construction.

Theorem ([5],Theorem 3.1). *The map*

$$\psi : Z \to eH_{0,\mathbf{c}}e$$

$$z \mapsto eze$$

is a Poisson isomorphism.

Let $\mathcal{A}$ denote the filtration on Z induced from that on H. The map ψ preserves the filtrations $\mathcal{A}$ and $\mathcal{B}$ and we have the following result.

Proposition ([5], Proposition 3.4). *The associated graded map* $\text{gr}(\psi) : \text{gr}^{\mathcal{A}}Z \to \text{gr}^{\mathcal{B}}eH_{0,\mathbf{c}}e$ *is an algebra isomorphism.*

Let P be a prime ideal of Z, then by the Theorem, P is Poisson if and only if $\psi(P)$ is Poisson. Furthermore, by the Proposition, $\mathcal{V}(\text{gr}^{\mathcal{A}}P)$ is irreducible if and only if $\mathcal{V}(\text{gr}^{\mathcal{B}}\psi(P))$ is irreducible.

Corollary 4. *Let I be a Poisson prime ideal of Z. Then $\mathcal{V}(\text{gr}^{\mathcal{A}}I)$ is irreducible.*

Proof. This is immediate from the previous paragraph and Corollary 3. $\qquad\square$

4. Preliminaries to the proof of theorem 2.3

4.1.   For the remainder of the paper retain the following notation: Let A, $\mathcal{F}$, $\langle -, - \rangle$, I, X and V satisfy all of the hypotheses of Theorem 2.3 and let $M = A/I$. We can choose an irreducible component of V of maximal dimension. As explained in 2.1, because X has FMSL, there exists a symplectic leaf $\mathcal{S}$ such that $\overline{\mathcal{S}}$ is this component. Then $\dim\mathcal{S} = \dim\overline{\mathcal{S}} = n$, and by definition, the closed points of $\mathcal{S}$ all have rank n. For a subvariety W of V we write sm$W$ for the smooth locus of W and sing$W$ for $W \setminus \text{sm}W$.

Lemma. (1) $\mathcal{S}$ *is open in* V.

(2) $dim\, \overline{\mathcal{S}} \setminus \mathcal{S} \leqslant \dim\overline{\mathcal{S}} - 2 = \dim V - 2.$

(3) $\mathcal{S}$ *is a homogeneous subvariety of* V *i.e. there exist homogeneous elements* $g_1, \ldots, g_s \in \mathrm{gr}^{\mathcal{F}} A / \mathrm{gr}^{\mathcal{F}} I$ *such that* $\mathcal{S} = V \setminus \mathcal{V}(g_1, \ldots, g_s).$

Proof. (1) Let $V = I_1 \cup \cdots \cup I_k$ be an irredundant irreducible decomposition of V with $I_1 = \overline{\mathcal{S}}$. We claim that $\mathcal{S} \cap I_j = \emptyset$ for all $2 \leqslant j \leqslant k$. If not, then for some j, $\mathcal{S} \cap I_j$ contains a closed point of rank n, $\mathbf{m}$ say. By [2, Proposition 3.7] the smooth locus of I_j, $\mathrm{sm} I_j$, is a symplectic leaf in V which contains $\mathbf{m}$. Then $\mathcal{S} = \mathrm{sm} I_j$ implies that $I_1 = I_j$, a contradiction.

Therefore $V \setminus \mathcal{S} = (\overline{\mathcal{S}} \setminus \mathcal{S}) \cup I_2 \cup \cdots \cup I_k$ is closed in V.

(2) It is clear from the definition of symplectic leaves that they are even dimensional. We can write $\overline{\mathcal{S}} \setminus \mathcal{S}$ as a finite union of symplectic leaves, each of which has dimension less than $\mathcal{S}$. The inequality follows because of even dimensionality.

(3) This is true because $\mathcal{S} = \mathrm{sm} I_1$. Since V is homogeneous I_1 is also homogeneous so we may assume that the ideal of I_1 is generated by homogeneous elements $h_1, \ldots, h_l$ in some polynomial ring $\mathbb{C}[x_1, \ldots x_m]$. Now $\mathrm{sing} I_1$ is defined as the points vanishing at certain $(m - r) \times (m - r)$ minors of
$$\left(\partial h_i / \partial x_j \right).$$
These minors are homogeneous polynomials in the $x_j s$. Hence $\mathrm{sing} I_1$ and also $\mathrm{sm} I_1$ are homogeneous subvarieties.

$\square$

(1) and (3) imply that there is a homogeneous open set $U \subseteq X$ such that $U \cap V = \mathcal{S}$. We write $U = X \setminus \mathcal{V}(f_1, \ldots, f_t)$ where the f_i are homogeneous elements of $\mathrm{gr}^{\mathcal{F}} A$.

4.2. **Microlocalisation.** The proof of the main theorem makes use of microlocalisation techniques which are described in [11, § 3-4]. We shall say that a $\mathbb{Z}$-filtration, $\mathcal{B}$, of an A-module M is *good* if $\bigcap_{n \in \mathbb{Z}} \mathcal{B}_n = 0$, $\bigcup_{n \in \mathbb{Z}} \mathcal{B}_n = M$ and $\mathrm{gr}^{\mathcal{B}} M$ is a finitely generated $\mathrm{gr}^{\mathcal{F}} A$-module. We recall the definition of support. Let R be a commutative ring and let N be an R-module. Then
$$\mathrm{supp}_R N = \{P \in \mathrm{Spec}\ R : N_P \neq 0\}.$$
We write supp N when the ring is clear from the context.

For $M = A/I$, consider the induced filtration on M (which we also call $\mathcal{F}$). Then
$$\mathrm{supp}\ \mathrm{gr}^{\mathcal{F}} M = \mathcal{V}(\mathrm{ann}\ \mathrm{gr}^{\mathcal{F}} M) = \mathcal{V}(\mathrm{gr}^{\mathcal{F}} I) = V$$
where the first equality is true because $\mathcal{F}$ is a good filtration of M. Now $\mathrm{gr}^{\mathcal{F}} M$ defines a sheaf of $\mathcal{O}_X$-modules, $\mathcal{M}$ on X. We only need to know the sections of $\mathcal{M}$ over U, which we can calculate explicitly:

$$\mathcal{M}(U) = \{(m_{f_i}) \in \prod_{i=1}^{t} (\mathrm{gr}^{\mathcal{F}} M)_{f_i} : m_{f_i} = m_{f_j} \in (\mathrm{gr}^{\mathcal{F}} M)_{f_i f_j} \forall i, j\}.$$

Lemma ([11], Lemma 3.3). *Let K be the kernel of the natural $\mathrm{gr}^{\mathcal{F}}A$-module map*

$$\beta : \mathrm{gr}^{\mathcal{F}}M \to \mathcal{M}(U)$$

$$m \mapsto (m).$$

Then $K = \{m \in \mathrm{gr}^{\mathcal{F}}M : \text{for each } i \text{ there exists } N_i \in \mathbb{N} \text{ such that } f_i{}^{N_i}m = 0\}$, and $\mathrm{supp}\, K \cap U = \emptyset$.

Microlocalisation introduces a new filtration, Γ, on M which is compatible with $\mathcal{F}$ [11, Corollary 6.9]. We can give a description of Γ in terms of the f_is introduced above. Let p_i be the degree of f_i and for each i choose a lift, ϕ_i, of f_i to A. Thus $\phi_i \in \mathcal{F}_{p_i}$ and $\sigma_{p_i}(\phi_i) = f_i$.

Let $\mathcal{I} = \{1, \ldots, t\}$ and suppose $\tau = (i_1, \ldots, i_N) \in \mathcal{I}^N$ is an ordered N-tuple of elements of $\mathcal{I}$. Define

$$p_\tau = \sum_{j=1}^{N} p_{i_j}, \quad \phi_\tau = \prod_{j=1}^{N} \phi_{i_j} \in \mathcal{F}_{p_\tau}.$$

Then $\Gamma_n = \{m \in M : \text{ for all } N \text{ sufficiently large, and for all } \tau \in \mathcal{I}^N, \phi_\tau \cdot m \in \mathcal{F}_{n+p_\tau}\}$. We see Γ has the property that $\mathcal{F}_i \subseteq \Gamma_i \ \forall \ i \in \mathbb{Z}$ so that $\mathrm{gr}^{\Gamma}M$ is a $\mathrm{gr}^{\mathcal{F}}A$-module and there is a canonical map (of $\mathrm{gr}^{\mathcal{F}}A$-modules) $\alpha : \mathrm{gr}^{\mathcal{F}}M \to \mathrm{gr}^{\Gamma}M$.

Proposition ([11], Proposition 3.11). *There exists a map of $\mathrm{gr}^{\mathcal{F}}A$-modules $\theta : \mathrm{gr}^{\Gamma}M \to \mathcal{M}(U)$ which is injective and gives rise to the following commutative diagram:*

$$
\begin{array}{ccc}
\mathrm{gr}^{\mathcal{F}}M & \xrightarrow{\ \alpha\ } & \mathrm{gr}^{\Gamma}M \\
& \searrow{\scriptstyle \beta} & \downarrow{\scriptstyle \theta} \\
& & \mathcal{M}(U)
\end{array}
$$

4.3. For a commutative ring R and an R-module N we recall that the *associated primes* of N, written $\mathrm{Ass}N$, are the set of primes of R which are annihilators of elements of N. The following result will be key to the proof of Theorem 2.3: we will use it to prove that $\mathcal{M}(U)$ is a finitely generated $\mathrm{gr}^{\mathcal{F}}A$-module. In fact, we show later that if we take $R = \mathrm{gr}^{\mathcal{F}}A$, $N = \mathrm{gr}^{\mathcal{F}}M$ and $W = U$ in the statement below then condition (4.1) is a consequence of Lemma 4.1 (2).

Theorem ([7], Proposition 5.11.1). *Suppose R is an affine commutative algebra over $\mathbb{C}$, N is a finitely generated R-module and W is an open set in $\mathrm{Spec}R$. Let $\mathcal{N}$ denote the sheaf of modules associated to N. Then the R-module $\mathcal{N}(W)$ is finitely generated if and only if for every prime $P \in W \cap \mathrm{Ass}N$, $\overline{P}$, the closure of P in $\mathrm{Spec}R$, satisfies*

$$\overline{P} \cap (\mathrm{Spec}R \backslash W) \text{ has codimension at least 2 in } \overline{P}. \tag{4.1}$$

We shall also require some information about the associated primes of $\mathrm{gr}^{\mathcal{F}}M$. Let R be a Poisson algebra and N an R-module, then we say that N is a *Poisson module* if there exists a $\mathbb{C}$-bilinear form $\{-, -\}_N : R \times N \to N$ satisfying $\{r, r'n\}_N = \{r, r'\}n + r'\{r, n\}_N$ for all $r, r' \in R$ and $n \in N$. It is clear that, in our setting, $\mathrm{gr}^{\mathcal{F}}M$ is a Poisson $\mathrm{gr}^{\mathcal{F}}A$-module.

Lemma ([3], Theorem 4.5). *Let R be a Noetherian Poisson algebra and N be a finitely generated Poisson R-module. Then the associated primes of N are Poisson ideals of R.*

Proof. Let P be an associated prime of N. Let $L = \{n \in N : P^i n = 0$ for some $i \geqslant 0\}$, this is a non-zero submodule of N. We claim that L is a Poisson submodule of N. To show this we first note that since L is finitely generated there is some $t \geqslant 1$ such that $P^t L = 0$. Let $l \in L, r \in R$. For all $r' \in P^t$,

$$0 = \{r, r'l\}_N = \{r, r'\}l + r'\{r, l\}_N.$$

Therefore $P^t \{R, L\}_N \subseteq \{R, P^t\}L \subseteq L$, which implies that $P^{2t}\{R, L\}_N = 0$. Hence $\{R, L\}_N \subseteq L$, by definition of L, which means that L is a Poisson submodule of N. By [2, Lemma 4.1], $\mathfrak{I} = \mathrm{ann}_R L$ is a Poisson ideal of R. There is some element $x \in L$ such that $\mathrm{Ann}_R\{x\} = P$, then $x \in L$ implies $\mathfrak{I} \subseteq P$. Taking radicals of the ideals $P^t \subseteq \mathfrak{I} \subseteq P$ yields $\mathrm{rad}\mathfrak{I} = P$, and therefore P is a Poisson ideal by [4, 3.3.2]. $\qquad\square$

5. Proof of Theorem 2.3

5.1. We retain the notation introduced at the beginning of section 4, in particular we recall that $M = A/I$. We also use the notation α, β and θ from Proposition 4.2.

We make the following two assumptions:

Claim (1). Γ is a good filtration of M.

Claim (2). $\mathrm{supp}_{\mathrm{gr}^{\mathcal{F}}A} \mathcal{M}(U) \subseteq \overline{\mathcal{S}}$.

Now $\mathrm{supp}_{\mathrm{gr}^{\mathcal{F}}A} \mathrm{gr}^{\Gamma}M \subseteq \mathrm{supp}_{\mathrm{gr}^{\mathcal{F}}A} \mathcal{M}(U)$ because θ is injective. The left hand side equals V by claim (1) and the right hand side is contained in $\overline{\mathcal{S}}$ by claim (2). So we have $V \subseteq \overline{\mathcal{S}}$ which implies that $V = \overline{\mathcal{S}}$ and this proves the theorem.

It remains to prove the two claims.

Proof of Claim (1): Recall that there are three conditions to check.

(a) $\bigcap_{n \in \mathbb{Z}} \Gamma_n = 0$. Let $M_{-\infty} = \bigcap_{n \in \mathbb{Z}} \Gamma_n$. It is easy to check that $M_{-\infty}$ is an A-sub-bimodule of M. We see that $\mathrm{gr}^{\Gamma}(M_{-\infty}) = 0$: for all $i \in \mathbb{Z}$, $(\Gamma_i \cap M_{-\infty})/(\Gamma_{i-1} \cap M_{-\infty}) = (\Gamma_{i-1} \cap M_{-\infty})/(\Gamma_{i-1} \cap M_{-\infty}) = 0$. Therefore the map $\mathrm{gr}^{\mathcal{F}}(M_{-\infty}) \to \mathrm{gr}^{\Gamma}(M_{-\infty})$ given by the restriction of the map α above, is the zero map. It follows from Proposition 4.2 that $\mathrm{gr}^{\mathcal{F}}(M_{-\infty}) \subseteq K$ where K is the kernel of β. By Lemma 4.2

$$\mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(M_{-\infty}) \cap U = \emptyset. \tag{5.1}$$

Now since $M_{-\infty}$ is an A-sub-bimodule of A/I, there is an ideal J of A such that $I \subseteq J \subseteq A$ and $M_{-\infty} = J/I$. Suppose that $M_{-\infty} \neq 0$. Then J properly contains I. It is a consequence of [9, Propositions 3.15 and 6.6] that $\dim\mathcal{V}(\mathrm{gr}^{\mathcal{F}}J) < \dim\mathcal{V}(\mathrm{gr}^{\mathcal{F}}I) = \dim\overline{\mathcal{S}}$. Let $\mathfrak{p}$ be the defining ideal of $\overline{\mathcal{S}}$. The equality of closed sets (where the support is considered over $\mathrm{gr}^{\mathcal{F}}A$)

$$\mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(A/I) = \mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(J/I) \cup \mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(A/J)$$

implies that $\mathfrak{p} \in \mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(J/I)$ and therefore that $\overline{\mathcal{S}} \subseteq \mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(J/I)$. Hence $\mathcal{S} \subseteq \overline{\mathcal{S}} \subseteq \mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(J/I) = \mathrm{supp}\,\mathrm{gr}^{\mathcal{F}}(M_{-\infty})$. This contradicts (5.1) and so $M_{-\infty} = 0$.

(b) $\bigcup_{n \in \mathbb{Z}} \Gamma_n = M$. This is straightforward because $\mathcal{F}_n \subseteq \Gamma_n$ implies $M = \bigcup_n \mathcal{F}_n \subseteq \bigcup_n \Gamma_n \subseteq M$.

(c) $\operatorname{gr}^\Gamma M$ *is a finitely generated* $\operatorname{gr}^\mathcal{F} A - module$. To prove this we in fact show that $\mathcal{M}(U)$ is a finitely generated $\operatorname{gr}^\mathcal{F} A$-module (which proves (c) by Proposition 4.2, since $\operatorname{gr}^\mathcal{F} A$ is Noetherian). We would like to show that $\mathcal{M}(U)$ is finitely generated and so by Theorem 4.3 it suffices to show that each prime $P \in U \cap \operatorname{Ass} \operatorname{gr}^\mathcal{F} M$ satisfies (4.1) with $R = \operatorname{gr}^\mathcal{F} A$. Let $P \in U \cap \operatorname{Ass} \operatorname{gr}^\mathcal{F} M \subseteq U \cap \operatorname{supp} \operatorname{gr}^\mathcal{F} M = \mathcal{S}$. By Lemma 4.3, P is a Poisson prime ideal of $\operatorname{gr}^\mathcal{F} A$. Now $U \cap \overline{P}$ is a nonempty open subset of $\overline{P}$ which means that it contains a closed point $\mathbf{m}$, and $U \cap \overline{P} \subseteq \mathcal{S}$ implies that $\mathbf{m}$ has rank n. We conclude that $\dim \overline{P} = n$ by [2, Lemma 3.1(5)], and therefore $\overline{P} = \overline{\mathcal{S}}$. Thus condition (4.1) is a consequence of Lemma 4.1 (2) and $\mathcal{M}(U)$ is finitely generated. This proves (c) and concludes the proof of Claim (1).

Proof of Claim (2): We must show that $P \notin \overline{\mathcal{S}} \Rightarrow \mathcal{M}(U)_P = 0$. Let $P \notin \overline{\mathcal{S}}$. This is equivalent to there being a neighbourhood, Y, of P in X such that $Y \cap \mathcal{S} = \emptyset$. Without loss of generality, we may assume that Y is some standard open set O_g for some $g \in \operatorname{gr}^\mathcal{F} A$ with $g \notin P$. We have $U = X \setminus \mathcal{V}(f_1, \ldots, f_t) = O_{f_1} \cup \cdots \cup O_{f_t}$; for each i, $O_{f_i g} = O_{f_i} \cap O_g$ is an open subset of U which intersects $\mathcal{S}$ trivially. Since $U \cap V = \mathcal{S}$ we conclude that $O_{f_i g}$ is contained in the open set $X \setminus V$, and therefore that $V \subseteq \mathcal{V}(f_i g)$. By considering the ideals of these subvarieties we deduce that $\operatorname{rad}\langle f_i g \rangle \subseteq \operatorname{rad}(\operatorname{ann} \operatorname{gr}^\mathcal{F} M)$. Hence there are integers k_i so that $(f_i g)^{k_i} \in \operatorname{ann} \operatorname{gr}^\mathcal{F} M$. We consider a typical element $(\frac{m_1}{f_1^{N_1}}, \ldots, \frac{m_t}{f_t^{N_t}}) \in \mathcal{M}(U)$. Let $k = max_i\{k_i\}$, then for all i:

$$(f_i g)^k m_i = 0 \Rightarrow \frac{g^k m_i}{f_i^{N_i}} = 0 \in \operatorname{gr}^\mathcal{F} M_{f_i} \Rightarrow \mathcal{M}(U)_P = 0$$

and this proves Claim (2).

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